lectures on functional analysis (2)

first course

The fourth stage

Mathematics Department

College of Education for Pure Sciences

Anbar University

المحاضرة الاولى

Example:

Every open and closed balls in normed space are Convex.

Sol. let X be normed space ; $x, y \in B_r(x_0)$ and $0 \leq \lambda \leq 1 \Rightarrow \|x - x_0\| < r$ and $\|y - x_0\| < r$

we need to prove $\lambda x + (1-\lambda) y \in B_r(x_0)$.

$\lambda x + (1-\lambda) - x_0 = \lambda(x - x_0) + (1-\lambda)(y - x_0)$

$$\Rightarrow \|\lambda x + (1-\lambda) y - x_0\| = \|\lambda(x - x_0) + (1-\lambda)(y - x_0)\|$$
$$\leq |\lambda| \|x - x_0\| + |(1-\lambda)| \|y - x_0\|$$
$$< |\lambda| r + |1-\lambda| r = r$$

because $|\lambda| = \lambda$ and $|1-\lambda| = 1-\lambda$, $\lambda, 1-\lambda > 0$

$\Rightarrow \lambda x + (1-\lambda) y \in B_r(x_0) \Rightarrow B_r(x_0)$ is Convex.

Similarly, we can to prove $\overline{B_r}(x_0)$ is Convex.

Theorem:

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in normed space X such that $x_n \to x$ and $y_n \to y$. Then

1. $x_n + y_n \to x + y$; 2. $\lambda x_n \to \lambda x$, $\forall \lambda \in F$.

3. $\|x_n\| \to \|x\|$ 4. $\|x_n - y_n\| \to \|x - y\|$

5. If $\{\lambda_n\}$ is a sequence in F such that $\lambda_n \to \lambda$, then $\lambda_n x_n \to \lambda x$.

Proof, (1) $\|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\|$

$\leq \| x_n - x \| + \| y_n - y \|$

since $\| x_n - x \| \to 0$ & $\| y_n - y \| \to 0$ as $n \to \infty$

So, $\| (x_n + y_n) - (x - y) \| \to 0$, as $n \to \infty$, i.e.

$x_n + y_n \to x + y$

(3) Since $| \, \|x_n\| - \|x\| \, | \leq \| x_n - x \|$ and $\|x_n - x\| \to 0$ as $n \to \infty$, i.e.

$\|x_n\| \to \|x\|$.

(4) $| \, \|x_n - y_n\| - \|x - y\| \, | \leq \| (x_n - y_n) - (x - y) \|$

$\leq \| x_n - \cancel{x_n} \| + \| y_n - y \|$

we have $\| x_n - x \| \to 0$ & $\| y_n - y \| \to 0$, as $n \to \infty$

Hence $| \, \| x_n - y_n \| - \| x - y \| \, | \to 0$ as $n \to \infty$.

(5) $\| \lambda_n x_n - \lambda x \| = \| \lambda_n x_n - \lambda_n x + \lambda_n x - \lambda x \|$

$= \| \lambda_n (x_n - x) + (\lambda_n - \lambda) x \|$

$\leq | \lambda_n | \, \| x_n - x \| + | \lambda_n - \lambda | \, \| x \|$

Since $\| x_n - x \| \to 0$ and $| \lambda_n - \lambda | \to 0$, as $n \to \infty$

$\Rightarrow \| \lambda_n x_n - \lambda x \| \to 0$, as $n \to \infty$.

Example: let $X = F$, we define the function $\| \cdot \| : X \to \mathbb{R}$ by $\|x\| = |x|$, $\forall x \in X$. show that X is Banach space.

Sol.: First, we need to show that X is a normed space

- since $|x| \geq 0$, for all $x \in X \Rightarrow \|x\| \geq 0$.
- let $x \in X \Rightarrow \|x\| = 0 \Leftrightarrow |x| = 0 \Leftrightarrow x = 0$.

- let $x \in X$ and $\alpha \in F \Rightarrow \|\alpha x\| = |\alpha x| = |\alpha||x|$ (33)
$= |\alpha| \|x\|$.

- let $x, y \in X$, $\|x+y\| = |x+y| \leq |x| + |y|$

$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$.

$\Rightarrow X$ is normed space.

since $\mathbb{R}$ or $\mathbb{C}$ is Complete space, F is Complete Space, hence X is Banach space.

Remark: let F^n denoted the set of all ordered n-tuples of elements in F of fixed $n \in \mathbb{N}$, i.e.

$F^n = \{x = (x_1, \ldots, x_n); \quad x_i \in F, i = 1, 2, \ldots, n\}$. Then F^n is a vector space under the following addition and multiplication by scalor

1. $x + y = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1+y_1, \ldots, y_n+x_n)$
for all $x, y \in F^n$.

2. $\lambda x = \lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$, $\forall x \in F^n$ and $\alpha \in F$

Example: let $X = F^n$, we define the function
$\|\cdot\|: X \to \mathbb{R}$ by $\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$ $\forall x \in F^n$.
Show that X is Banach space.

Sol.: First, we need to show that X is normed space

1. since $|x_i| \geq 0$ $\forall$ $1 \leq i \leq n \Rightarrow \|x\| \geq 0$

2. let $x \in X$, $\|x\| = 0 \Leftrightarrow \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} = 0 \Leftrightarrow x_i^2 = 0$

For $i = 1, 2, \dots, n$

So, $\|x\| = 0 \Leftrightarrow x_i = 0 \,,\ \forall\ i = 1, 2, \dots, n \Leftrightarrow x = 0$

3. let $x \in X$ and $\lambda \in F \Rightarrow \|\lambda x\| = \left(\sum_{i=1}^{n} |\lambda x_i|^2\right)^{1/2}$

$$= |\lambda| \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

$$= |\lambda|\, \|x\|$$

4. let $x, y \in X \Rightarrow \|x+y\| = \left(\sum_{i=1}^{n} |x_i + y_i|^2\right)^{1/2}$

By using Minkowski inequality $\leq \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} + \left(\sum_{i=1}^{n} |y_i|^2\right)^{2}$

$$= \|x| + \|y\|$$

$\Rightarrow X$ is a normed space.

second, to show that X is complete.

let $\{x_m\}$ be a cauchy sequence in X, $\exists\ k \in \mathbb{Z}^{+} \ni$

$\|x_m - x_l\| < \epsilon \,,\ \forall\ m, l > k$

$\Rightarrow \|x_m - x_l\|^2 < \epsilon^2 \quad \forall\ m, l > k$ --- (1

Since $x_m - x_l = (x_1^m - x_1^l, x_2^m - x_2^l, \dots, x_n^m - x_n^l)$

because $x_m \in F^n \Rightarrow x_m = (x_1^m, x_2^m, \dots, x_n^m)$

So, $\|x_m - x_l\|^2 = \sum_{i=1}^{n} |x_i^m - x_i^l|^2$ --- (2

From (1) and (2) $\sum_{i=1}^{n} |x_i^m - x_i^l|^2 < \epsilon^2 \quad \forall\ m, l > k$

$\Rightarrow |x_i^m - x_i^l| < \epsilon^2 \quad \forall\ m, l > k$

$\Rightarrow |x_i^m - x_i^l| < \epsilon \quad \forall\ m, l > k$

So that for each i, the sequence $\{x_m\}$ is cauchy sequence in F.

المحاضرة الثانية

Since F is complete, then for each i, the sequence $\{x_m\}$ is converges to a point, say $x_i \in F$

$\Rightarrow x_m \rightarrow x_i \quad \forall \quad 1 \leq i \leq n$

put $x = (x_1, \dots, x_n) \Rightarrow x \in F^n$

we must prove $x_m \rightarrow x$

let $\epsilon > 0$, for all $m > k$, we have $\| x_m - x \|^2 = \sum_{i=1}^{n} | x_i^m - x_i |^2$

$< \epsilon^2$

$\forall m, l > K$

$\Rightarrow \| x_m - x \| < \epsilon \quad \forall m, > K$

$\Rightarrow \{x_m\}$ is converges to $x \in F^n = X$

$\Rightarrow X$ is complete $\Rightarrow X$ is Banach space.

Example: let $X = \mathbb{R}^n$, we define the function

$\|\cdot\| : X \rightarrow \mathbb{R}$ by $\|x\| = \sum_{i=1}^{n} |x_i|$.

for all $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Show that X is Banach space.

Sol: First, we need to show that X is normed space.

1. since $|x_i| \geq 0 \quad \forall \quad 1 \leq i \leq n \Rightarrow \|x\| \geq 0$

2. let $x \in X$, $\|x\| = 0 \Leftrightarrow \sum_{i=1}^{n} |x_i| = 0$

$\Leftrightarrow |x_i| = 0, \forall \ 1 \leq i \leq n$

$\Leftrightarrow x_i = 0, \forall \ 1 \leq i \leq n$

$\Leftrightarrow x = 0$.

3. let $x \in X$ and $\lambda \in \mathbb{R}$

$\lambda x = \lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$

$\Rightarrow \quad \|\lambda x\| = \sum_{i=1}^{n} |\lambda x_i| = |\lambda| \sum_{i=1}^{n} |x_i| = |\lambda| \, \|x\|.$

4. let $x, y \in X$

$$\|x+y\| = \sum_{i=1}^{n} |x_i + y_i| \le \sum_{i=1}^{n} (|x_i| + |y_i|)$$

$$= \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i|$$

$\Rightarrow \quad \|x+y\| \le \|x\| + \|y\|.$

$\Rightarrow \quad X$ is normed space

second, since $X = \mathbb{R}$ is Complete space

$\Rightarrow \quad X$ is Banach space.

Example: let $X = \mathbb{R}^n$, we define the function $\|\cdot\| : X \longrightarrow \mathbb{R}$ by $\|x\| = \max\{|x_1|, |x_2|, \dots, |x_n|\}$ for all $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Then X is Banach space.

Remark: let $C[a,b]$ be the set of all real-valued bounded continuous function, defined on $[a,b]$, i.e. $f \in C[a,b]$ iff $f : [a,b] \longrightarrow \mathbb{R}$ is bounded and cont^s. function. Then $C[a,b]$ is vector space under the following addition and scalar multiplication

1. $(f+g)(x) = f(x) + g(x), \ \forall f, g \in C[a,b].$
2. $(\alpha f)(x) = \alpha f(x), \ \forall f \in C[a,b], \alpha \in F.$

Example: let $X = C[a,b]$, we define the function $\|\cdot\| : X \to \mathbb{R}$ by $\|f\| = \max\{|f(x)| ; a \le x \le b\}$ for all $f \in X$. Prove that X is normed space.

Sol. First, we need to show that is a normed space

1. Since $|f(x)| \geq 0$, $\forall x \in [a,b] \Rightarrow \|f\| \geq 0$.

2. $$\|f\| = 0 \iff \max\{|f(x)| ; a \le x \le b\} = 0$$
$$\iff |f(x)| = 0 \quad , x \in [a,b]$$
$$\iff f(x) = 0 \quad , x \in [a,b]$$
$$\iff f = 0$$

3. let $f \in X$ and $\alpha \in \mathbb{R}$
$$\|\alpha f\| = \max\{|(\alpha f)(x)| ; a \le x \le b\}$$
$$= \max\{|\alpha f(x)| ; a \le x \le b\}$$
$$= \max\{|\alpha|\,|f(x)| ; a \le x \le b\}$$
$$= |\alpha| \max\{|f(x)| ; a \le x \le b\}$$
$$= |\alpha|\,\|f\|.$$

4. $$\|f+g\| = \max\{|(f+g)(x)| ; a \le x \le b\}$$
$$= \max\{|f(x)+g(x)| ; a \le x \le b\}$$
$$\le \max\{|f(x)| + |g(x)| ; a \le x \le b\}$$
$$= \max\{|f(x)| ; a \le x \le b\} + \max\{|g(x)| ; a \le x \le b\}$$
$$= \|f\| + \|g\|.$$
whenever $f, g \in X$.

$\Rightarrow X$ is normed space

Example: let $X = C[0,1]$, we define the function $\|\cdot\| : X \to \mathbb{R}$, by $\|f\| = \int_0^1 |f(x)|\,dx$ for all $f \in X$, $x \in [0,1]$. Show that X is normed space, but not Banach space.

<u>Sol.</u> First, to show X is a normed space.

1. Since $|f(x)| \geq 0$ for all $x \in [0,1] \Rightarrow \|f\| \geq 0$.

2. (i) If $f = 0$, then $\int_0^1 |f(x)|\,dx = 0 = \|f\|$.

(ii) If $\|f\| = 0$, then $\int_0^1 |f(x)|\,dx = 0$

Since $|f(x)| \geq 0$ and f is cont^s., then $|f(x)| = 0 \Rightarrow f = 0$.

3. let $f \in X$ and $\alpha \in \mathbb{R}$

$$\|\alpha f\| = \int_0^1 |(\alpha f)(x)|\,dx = \int_0^1 |\alpha f(x)|\,dx$$

$$= \int_0^1 |\alpha|\,|f(x)|\,dx = |\alpha| \int_0^1 |f(x)|\,dx$$

$$= |\alpha|\,\|f\|.$$

4. Let $f, g \in X$

$$\|f+g\| = \int_0^1 |(f+g)(x)|\,dx = \int_0^1 |f(x)+g(x)|\,dx$$

$$\leq \int_0^1 (|f(x)| + |g(x)|)\,dx$$

$$= \int_0^1 |f(x)|\,dx + \int_0^1 |g(x)|\,dx$$

$$= \|f\| + \|g\|.$$

$\Rightarrow X$ is normed space.

We now that X is not complete.

المحاضرة الثالثة

Consider the sequence $\{f_n\}$ in X defined as follows (39)

$$f_n(x) = \begin{cases} 1, & 0 \le x \le \frac{1}{2} \\ -nx + \frac{1}{2}n + 1, & \frac{1}{2} < x \le \frac{1}{2} + \frac{1}{n} \\ 0, & \frac{1}{2} + \frac{1}{n} < x \le 1 \end{cases}$$

Then $\{f_n\}$ is a Cauchy sequence in X, because, if $m > n \geq 3$, then

$$\| f_m - f_n \| = \int_0^1 |(f_m - f_n)(x)| \, dx = \int_0^1 |f_m(x) - f_n(x)| \, dx$$

$$= \int_0^{\frac{1}{2}} |1-1| \, dx + \int_{\frac{1}{2}}^1 |f_m(x) - f_n(x)| \, dx$$

$$\|f_m - f_n\| \le \int_{\frac{1}{2}}^1 |f_m(x)| \, dx + \int_{\frac{1}{2}}^1 |f_n(x)| \, dx$$

$$= \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{m}} |-mx + \tfrac{1}{2}m + 1| \, dx + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} |-nx + \tfrac{1}{2}n + 1| \, dx$$

Since $-mx + \frac{1}{2}m + 1 \geq 0$, when $\frac{1}{2} < x < \frac{1}{2} + \frac{1}{m}$

$$\| f_m - f_n \| \le \frac{1}{2m} + \frac{1}{2n} \Rightarrow \| f_m - f_n \| \to 0 \text{ as } n \to \infty, m \to \infty$$

$\Rightarrow \{f_n\}$ is Cauchy sequence. But this sequence is not Convergent in X.

For, if there existed a $f \in X$ such that $f_n \to f$

$$\Rightarrow f(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2} \\ 0, & \frac{1}{2} < x \le 1 \end{cases}$$

. This condradiction because f is not Cont^s.

Theorem: let M be a subspace of Banach space X, Then M is Banach space if and only if it is closed in X.

Proof: suppose M is a Banach space and to prove that M is closed i.e. $M = \overline{M}$

we have always, $M \subset \overline{M}$ --- (1)

let $x \in \overline{M}$, there is a sequence $\{x_n\}$ in M such that $x_n \longrightarrow x$

$\Rightarrow$ $\{x_n\}$ is Cauchy sequence in M

we have M is Complete

$\Rightarrow$ $x_n \longrightarrow x \in M$, because the Convergent Point is a unique.

$\Rightarrow$ $\overline{M} \subset M$ -- (2)

From (1) and (2), we get $M = \overline{M} \Rightarrow M$ is closed.

Conversely, Assume that M is closed set in X.

let $\{x_n\}$ be a cauchy sequence in M.

Since $M \subset X \Rightarrow$ $\{x_n\}$ is a cauchy sequence in X.

Since X is Complet space, there is $x \in X \ni x_n \longrightarrow x$

So, $x_n \in M \Rightarrow x \in \overline{M}$, M is closed i.e. $M = \overline{M}$

$\Rightarrow$ $x \in M$

$\Rightarrow$ $\{x_n\}$ is Cauchy sequence Converge in M

$\Rightarrow$ M is Complete $\Rightarrow$ M is Banach space.

Theorem: Every finite dimensional normed space is Complete.

Proof: let X be a finite dimensional normed space with $\dim X = n > 0$ and let $\{e_1, e_2, \dots, e_n\}$ be a basis for X, take $\{x_m\}$ any Cauchy sequence in X i.e.

$$\|x_m - x_k\| \longrightarrow 0 \quad \text{as } m, k \longrightarrow \infty \quad \text{---(1)}$$

Since $x_m, x_k \in X \Rightarrow x_m = \sum_{i=1}^{n} \overset{(m)}{\alpha_i} e_i$; $\overset{m}{\alpha_i} \in F$

also $x_k = \sum_{i=1}^{n} \overset{k}{\alpha_i} e_i$, $\overset{k}{\alpha_i} \in F$

$$\Rightarrow x_m - x_k = \sum_{i=1}^{n} (\overset{m}{\alpha_i} - \overset{k}{\alpha_i}) e_i$$

Since $\{e_1, \dots, e_n\}$ is linear independent, by Lemma of linear Combination, there is $c > 0$ such that

$$\|x_m - x_k\| = \left\| \sum_{i=1}^{n} (\overset{m}{\alpha_i} - \overset{k}{\alpha_i}) e_i \right\| \geq c \sum_{i=1}^{n} |\overset{m}{\alpha_i} - \overset{k}{\alpha_i}| \quad \text{---(2)}$$

From (1) and (2), we have $\sum_{i=1}^{n} |\overset{m}{\alpha_i} - \overset{k}{\alpha_i}| \longrightarrow 0$ as $m, k \longrightarrow \infty$ for $i = 1, 2, \dots, n$

$\Rightarrow |\overset{m}{\alpha_i} - \overset{k}{\alpha_i}| \longrightarrow 0$ as $m, k \longrightarrow \infty$ for $i = 1, 2, \dots, n$

For $i = 1, 2, \dots, n \Rightarrow \{\overset{m}{\alpha_i}\}$ is Cauchy sequence in F.

Since F is either $\mathbb{R}$ or $\mathbb{C}$ and each for $\mathbb{R}$ or $\mathbb{C}$ are Complete $\Rightarrow \exists\, \alpha_i \in F \ni \overset{m}{\alpha_i} \longrightarrow \alpha_i$.

Put $x = \sum_{i=1}^{n} \alpha_i e_i \Rightarrow x_m \longrightarrow x$, $x \in X$

$\Rightarrow X$ is complete.

Corollary: Every finite dimensional subspace M of a normed space X is closed.

Proof: Since M is a finite dimensional subspace of a normed space $X \Longrightarrow M$ is Complet space $\Longrightarrow M$ is closed

Note: The infinite dimensional subspace of Banach space need not be closed.

Definition: let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a vector space X. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent (or $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$), written $\|\cdot\|_1 \cong \|\cdot\|_2$ if there exist positive real numbers a and b such that $a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$ for all $x \in X$.

Example: let $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ and $\|x\|_2 = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$ for all $x \in \mathbb{R}^n$. Show that $\|\cdot\|_1 \cong \|\cdot\|_2$

Sol. From Cauchy's inequality, we have $\sum_{i=1}^{n} |x_i y_i| \leq (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} (\sum_{i=1}^{n} y_i^2)^{\frac{1}{2}}$

for all $x_i, y_i \in \mathbb{R}$, $1 \leq i \leq n$

put $y_i = 1$ for all $i = 1, 2, \ldots, n$, we have

$$\sum_{i=1}^{n} |x_i| \leq (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} \cdot (\sum_{i=1}^{n} 1)^{\frac{1}{2}}$$

So, $\|x\|_1 \leq \|x\|_2 \cdot \sqrt{n} \Rightarrow \frac{1}{\sqrt{n}} \|x\|_1 \leq \|x\|_2$

set $a = \frac{1}{\sqrt{n}}$ and $b = 1$, we have $\|x\|_2 \leq \|x\|_1$.

Hence $\|\cdot\|_1$ & $\|\cdot\|_2$ are equivalent i.e.

$\|\cdot\|_1 \cong \|\cdot\|_2$.

المحاضرة الرابعة

Lemma (Linear Combination)

Let $\{x_1, x_2, \dots, x_n\}$ be a linear independent set of vectors in normed space X. Then there is a number $c > 0$, such that $\| \sum_{i=1}^{n} \alpha_i x_i \| \geq c \sum_{i=1}^{n} |\alpha_i|$, for all $\alpha_i \in F$, $i = 1, 2, \dots, n$.

Theorem: On a finite dimensional vector space all norms are equivalent.

Proof: let X be a finite dimensional vector space with $\dim X = n > 0$ and $\|\cdot\|_1$ & $\|\cdot\|_2$ be two norms on X.

To prove $\|\cdot\|_1 \cong \|\cdot\|_2$.

let $\{e_1, e_2, \dots, e_n\}$ be a basis for $X \Rightarrow \forall x \in X$ has a unique representation $x = \sum_{i=1}^{n} \alpha_i e_i$, $\alpha_i \in F$

$$\|x\|_1 = \| \sum_{i=1}^{n} \alpha_i e_i \|_1 \leq \sum_{i=1}^{n} |\alpha_i| \, \|e_i\|_1$$

$$\leq \max \|e_i\|_1 \sum_{i=1}^{n} |\alpha_i|$$

set $\max \|e_i\|_1 = k \Rightarrow \|x\|_1 \leq k \sum_{i=1}^{n} |\alpha_i| \quad \dots (1)$

since $\{e_1, e_2, \dots, e_n\}$ is basis for X and using L.C Lemma, $\exists\, c > 0, \ni \| \sum_{i=1}^{n} \alpha_i e_i \|_2 \geq c \sum_{i=1}^{n} |\alpha_i|$

$$\Rightarrow \|x\|_2 \geq c \sum_{i=1}^{n} |\alpha_i| \quad \dots (2)$$

From (1) and (2), we get

$$\frac{k}{c} \|x\|_1 \leq \sum_{i=1}^{n} |\alpha_i| \leq \|x\|_2$$

$$\Rightarrow \frac{k}{c} \|x\|_1 \leq \|x\|_2 \quad \dots (3)$$

Similarly, we can

$$\|x\|_2 \le \frac{c}{k} \|x\|_1 \quad --- (4)$$

From (3) and (4), we obtain

$$a = \frac{k}{c}\|x\|_1 \le \|x\|_2 \le b = \frac{c}{k}\|x\|_1 .$$

$$\Rightarrow \quad \|\cdot\|_1 \cong \|\cdot\|_2 .$$

Continuity

Def. let X and Y be two normed spaces. A function $f: X \longrightarrow Y$ is called

1. Continuous at $x_0 \in X$, if for each $\epsilon > 0$, there is $\delta > 0$ such that $x \in X$, $\|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \epsilon$.
   or equivalently, a function f is continuous at $x_0 \in X$, if for every sequence $\{x_n\}$ in X converging to x_0, the sequence $\{f(x_n)\}$ in Y converges to $f(x_0) \in Y$ i.e.

   $$x_n \longrightarrow x_0 \Rightarrow f(x_n) \longrightarrow f(x_0).$$

2. Compact if $f(X)$ contained in compact subset of Y.
3. Completely continuous if it is both continuous and compact.
4. Finite dimensional if it is compact function and $f(X)$ contained in a finite dimensional subspace of Y.

Theorem:

let X be a normed space. Then the function $f: X \to \mathbb{R}$, $f(x) = \|x\|$ is Continuous, the norm $\|\cdot\|$ on X is Cont'. function.

Proof: let $x_0 \in X$ and $\{x_n\}$ seq. in X $\ni$ $x_n \to x_0$ as $n \to \infty$.

Now,
$$|f(x_n) - f(x_0)| = |\|x_n\| - \|x_0\||$$
$$\leq \|x_n - x_0\|$$

since $x_n \to x_0 \Rightarrow \|x_n - x_0\| \to 0$ as $n \to \infty$

$\Rightarrow |f(x_n) - f(x_0)| \to 0$ as $n \to \infty$

$\Rightarrow f(x_n) \to f(x_0)$

$\Rightarrow f$ is cont'. at x_0 and x_0 is arbitrary point.

$\Rightarrow f$ is Cont' on X.

Theorem: let X be a normed space. Then functions $f: X \times X \to X$, $f(x,y) = x+y$ & $g: F \times X \to X$, $g(\alpha, x) = \alpha x$ are Continuous, in other words vector addition and scalar multiplication are jointly Continuous.

Proof: let $x_0, y_0 \in X$ and $\{x_n\}, \{y_n\}$ in X such that $x_n \to x_0$ & $y_n \to y_0$ as $n \to \infty$.

Now: $\|f(x_n, y_n) - f(x_0, y_0)\| = \|(x_n + y_n) - (x_0 + y_0)\|$

$= \|(x_n - x_0) + (y_n - y_0)\|$

$\leq \|x_n - x_0\| + \|y_n - y_0\|$

$\leq \to 0 + \to 0 = 0$, as $n \to \infty$

$\Rightarrow f(x_n, y_n) \to f(x_0, y_0)$ as $n \to \infty$

$\Rightarrow f$ is continuous at (x_0, y_0) and (x_0, y_0) is any point in $X \times X$, hance f is continuous.

Also, let $x_0 \in X$, $\alpha \in F$ and $\{x_n\}$ in X, $\{\alpha_n\}$ in F such that $x_n \to x_0$ and $\alpha_n \to \alpha$ as $n \to \infty$

Now, $\|g(\alpha_n, x_n) - g(\alpha, x_0)\| = \|\alpha_n x_n - \alpha x_0\|$

$= \|\alpha_n x_n - \alpha_n x_0 + \alpha_n x_0 - \alpha x_0\|$

$= \|\alpha_n (x_n - x_0) + (\alpha_n - \alpha) x_0\|$

$\leq |\alpha_n| \, \|x_n - x_0\| + |\alpha_n - \alpha| \, \|x_0\|$

Since $\|x_n - x_0\| \to 0$ and $|\alpha_n - \alpha| \to 0$ as $n \to \infty$, we have $\|g(\alpha_n, x_n) - g(\alpha, x_0)\| \to 0$ as $n \to \infty$

$g(\alpha_n, x_n) \to g(\alpha, x_0)$, as $n \to \infty$.

g is continuous at (α, x_0) and (α, x_0) is any point in $F \times X$, hence g is continuous.

Corollary:

Every normed space X is topological linear space.

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Continuous Linear Functions:

Recall that a function $f: X \to Y$ from a linear space X into linear space Y is called a linear if:

$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$, for all $x, y \in X$ and $\alpha, \beta \in F$.

Remarks:

i. Linear function of linear space X into its field F is called linear functional on X.

ii. let $L(X, Y)$ denoted the set of all linear functions from a linear space X into a linear space Y. Then $L(X, Y)$ is a vector space under the following addition and scalar multiplication

1. $(f+g)(x) = f(x) + g(x)$, $\forall f, g \in L(X, Y)$.

2. $(\alpha f)(x) = \alpha f(x)$. $\forall f \in L(X, Y)$ and $\alpha \in F$.

If $Y = X$, we write $L(X)$ instead of $L(X, X)$. The space of all linear functionals defined on a linear space X is called the algebraic dual space and denoted by X', i.e. $X' = L(X, F)$.

3. We say that X, Y are linear isomorphic (we write $X \cong Y$), the there is a bijective linear function $f: X \to Y$ such function is called linear isomorphism.

Theorem:

let X be a linear space over a field F.

1. If $x \in X$ and a function $T_x : X' \longrightarrow F$ defined by $T_x(f) = f(x)$, for all $f \in X'$, then T_x is linear functional i.e. $T_x \in X''$ and it is called Evaluation functional induced by x.

2. If the function $\psi : X \longrightarrow X''$ defined by $\psi(x) = T_x$ for all $x \in X$, then ψ injection linear function and ψ is called Canonical function.

Proof: (1) let $f, g \in X'$, $\alpha, \beta \in F$

$$T_x(\alpha f + \beta g) = (\alpha f + \beta g)(x) = (\alpha f)(x) + (\beta g)(x).$$
$$= \alpha f(x) + \beta g(x) = \alpha T_x(f) + \beta T_x(g).$$

$$\Rightarrow T_x \in X''.$$

(2) let $x, y \in X$, $\alpha, \beta \in F$

$$\Rightarrow \psi(\alpha x + \beta y) = T_{\alpha x + \beta y}$$

for all $f \in X'$, $T_{\alpha x + \beta y}(f) = f(\alpha x + \beta y)$

$$= \alpha f(x) + \beta f(y)$$
$$= \alpha T_x(f) + \beta T_y(f)$$
$$= (\alpha T_x + \beta T_y)(f)$$

$$\Rightarrow \psi(\alpha x + \beta y) = \alpha T_x + \beta T_y = \alpha \psi(x) + \beta \psi(y).$$

$\Rightarrow \psi$ is linear function.

Now, to prove that ψ is injection, let $x, y \in X$ such that $\psi(x) = \psi(y)$

$\Rightarrow T_x = T_y \Rightarrow T_x(f) = T_y(f)$ for $f \in X'$

$\Rightarrow f(x) = f(y)$ for all $f \in X'$

$\Rightarrow f(x-y) = 0$ for all $f \in X'$

$\Rightarrow x - y = 0 \Rightarrow x = y \Rightarrow \varphi$ is injective.

<u>Definition</u>: let X be a linear space over a field F. we say that X is an Algebrically Reflexive if φ is an onto, where φ is defined above theorem.

Theorem: Every finite dimensional space is algebraically reflexive.

Proof: let X be a finite dimensional space over a field F. $\Rightarrow \dim X' = \dim X$, so that X' finite dimensional. $\Rightarrow \dim X'' = \dim X$, so that X'' finite dimensional.

Since $\varphi : X \longrightarrow X''$ is injective and X', X'' are finite dimensional and $\dim X'' = \dim X$, then φ is onto.

Remark:

Recall that a function f from a topological space X into topological space Y, i.e. $f : X \longrightarrow Y$ is called continuous at a point $x \in X$ if every neighborhood U of $f(x)$ in Y there is a neighborhood V of x in $X \ni f(V) \subset U$. If f is continuous at every point, it is called continuous. A function $f : X \longrightarrow Y$ is cont iff each open (resp. closed) set U in Y the set $f^{-1}(U)$ is open (resp. closed) set in X.

Def. let (X, d) and (Y, d^*) be metric spaces.

A function $f: X \longrightarrow Y$ is called an Isometry if
(1) f is bijective.
(2) $d^*(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.

Def.: let X and Y be a normed spaces. An isometric isomorphism of X into Y is a one-one linear function f of X into Y such that $\|f(x)\| = \|x\|$ for every $x \in X$. Also we say that X is isametrically isomorphic or (congruent) to Y if there exists an isomorphism of X onto Y.

Remark: let f be an isometric isomorphism of X into Y where X and Y are normed spaces. let $x, y \in X$. Then $\|f(x) - f(y)\| = \|f(x-y)\| = \|x-y\|$. Thus f preserves distances and so it is an isometry.

Def. let X and Y be normed spaces. A topological Isomorphism of X into Y is a 1-1 linear function f of X into Y such that f and f^{-1} are continuous on their respective domains. Also we say that X is topologi-cal isomorphic to Y if there exists a topological isomorph-ism of X into Y. In other words, X and Y are topol-ogically isomorphic provided there exists a homeomorphis-m of X onto which is also a linear function.

Remark: Topological Isomorphism space need not be isometrically isomorphic. In fact there do exists examples of pairs spaces which are topologically isomorphic but not congruent.

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Theorem:
let X and Y be a normed spaces. Then X and Y are topologically isomorphic iff there exists a linear function of X onto Y and positive constant α, β such that $\alpha \|x\| \leq \|f(x)\| \leq \beta \|x\|$.

Proof: suppose X and Y are topologically isomorphic, then there exist a linear function f of X onto Y such that f and f^{-1} are continuous.
since f is cont' iff there exists a positive constant α such that $\|f(x)\| \leq \alpha \|x\|$ for all $x \in X$.
Again f^{-1} is cont'. iff there exists a positive constant β such that $\beta \|x\| \leq \|f(x)\|$, for all $x \in X$.
It follows that X and Y are topologically isomorphic iff there exists a linear function of X onto Y and positive constants α and β such that $\beta \|x\| \leq \|f(x)\| \leq \alpha \|x\|$.

Theorem: let X and Y be topological linear spaces and let $f: X \longrightarrow Y$ be a linear function. If f is cont' at 0, then it is continuous.

proof: let $x \in X$ and U be neighborhood of $f(x)$ in Y.
Then $U = f(x) + W$, where W is neighborhood of 0 in Y.
Since f is cont' at 0 in X, then there exist a neighbor. V of 0 in X such that $f(V) \subset W \Rightarrow x+V$ is neighbor. of x in X.
To show that $f(x+V) \subset U$.

let $z \in f(x+V) \Rightarrow \exists\, y \in x+V \ni f(y) = z$

Since $y \in x+V \Rightarrow y-x \in V \Rightarrow f(y-x) \in f(V)$

$\Rightarrow f(y) - f(x) \in f(V) \Rightarrow z - f(x) \in f(V)$

$\Rightarrow z \in f(x) + f(V) \Rightarrow z \in U$

$\Rightarrow f(x+V) \subset U$

$\Rightarrow f$ is cont'. at x, x is arbitrary point

$\Rightarrow f$ is cont'.

Theorem: let X and Y be a normed spaces and let $f: X \to Y$ be a linear function. Then f is cont's. either at every point of X or no point of X.

Proof: let x_1 and x_2 be any two points of X and suppose f is cont's. at x_1. Then to each $\epsilon > 0$, there exists $\delta > 0 \ni \|x - x_1\| < \delta \Rightarrow \|f(x) - f(x_1)\| < \epsilon$.

Now, $\|x - x_2\| < \delta \Rightarrow \|(x + x_1 - x_2) - x_1\| < \delta$

$\Rightarrow \|f(x + x_1 - x_2) - f(x_1)\| < \epsilon$

$\Rightarrow \|f(x) + f(x_1) - f(x_2) - f(x_1)\| < \epsilon$

$\Rightarrow \|f(x) - f(x_2)\| < \epsilon$

$\Rightarrow f$ is cont' at x_2, then f is cont's..

Theorem: let X and Y be a Banach spaces. If $f: X \to Y$ is cont's, linear and onto function, then f is open.

Proof: let G be open set in X. We want to show that $f(G)$ is open in Y.

let $y \in f(G)$, then $y = f(x)$ for some $x \in G$
Since G is open set in X, there is $r > 0 \ni B_r(x) \subset G$
$\Rightarrow f(B_r(x)) \subset f(G)$.
Since $B_r(x) = x + B_r(0) \Rightarrow x + B_r(0) \subset G$
By Lemma, there is an open sphere $B'_r(0)$ in Y center at origin such that $B'_r(0) \subset f(B_r(0))$

$$\Rightarrow y + B'_r(0) \subseteq y + f(B_r(0)) = f(x) + f(B_r(0))$$
$$= f(x + B_r(0)) = f(B_r(x)) \subset f(G)$$

since $y + B'_r(0) = B'_r(y) \Rightarrow B'_r(y) \subset f(G)$
$\Rightarrow f(G)$ is open, thus f is an open.

Def.: let X and Y be any non-empty sets and let $f: X \to Y$ be a function. The set

$$\{(x,y) \in X \times Y,\ y = f(x)\} = \{(x, f(x))\ ;\ x \in X, f(x) \in Y\}$$

is called the graph of f. We shall denote the graph of f by f_G. i.e.

$$f_G = \{(x,y) \in X \times Y,\ y = f(x)\} = \{(x, f(x))\ ;\ x \in X,\ f(x) \in Y\}.$$

In the case X and Y are normed spaces. Then $X \times Y$ is normed spaces. We will now generalize the above notion of graph.

Closed linear Functions

Def. let X and Y be a normed spaces and let D be a subspace of X. The linear function $f: D \to Y$ is called closed if every sequence $\{x_n\}$ in D such that $x_n \to x \in X$ and $f(x_n) \to y$, then $x \in D$ and $y = f(x)$.

Theorem: let X and Y be a normed spaces and let D be a subspace of X. The linear function $f: D \to Y$ is closed iff its graph f_G is closed subspace.

Proof: suppose that $f: D \to Y$ is closed and to prove that f_G is closed subspace.

let (x, y) be any limit point of f_G, i.e. $(x, y) \in \overline{f_G}$.

Then there is sequence of points in f_G, $(x_n, f(x_n))$ where $x_n \in D$ such that $(x_n, f(x_n)) \to (x, y)$

$\Rightarrow (x_n, f(x_n)) - (x, y) \to 0 \Rightarrow \|x_n - x\| \to 0$ and $\|f(x_n) - y\| \to 0$

$\Rightarrow x_n \to x$ and $f(x_n) \to y$

Since $f: D \to Y$ is closed, then $x \in D$ and $f(x) = y$

$\Rightarrow (x, y) \in f_G \Rightarrow f_G$ is closed.

Conversely, let the graph f_G is closed. To prove that the linear function $f: D \to Y$ is closed.

let $\{x_n\}$ be a sequence in D such that $x_n \to x \in X$ and $f(x_n) \to y \Rightarrow (x_n, f(x_n)) \to (x, y)$

$\Rightarrow (x, y) \in \overline{f_G}$ since f_G is closed $\Rightarrow \overline{f_G} = f_G$

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$\Rightarrow (x,y) \in f_G \Rightarrow x \in D$ and $y = f(x)$.

$\Rightarrow$ linear function $f: D \to Y$ is closed.

Theorem: let X and Y be a Banach spaces. If $f: X \to Y$ is a linear function, then f is cont^s. Iff its graph is closed.

Proof: suppose that f is cont^s. To prove f_G is closed let (x,y) be any limit point of f_G i.e. $(x,y) \in \bar{f}_G$

$\Rightarrow \exists (x_n, f(x_n))$ in $f_G \ni (x_n, f(x_n)) \to (x,y)$

$\Rightarrow (x_n, f(x_n)) - (x,y) \to 0 \Rightarrow \| (x_n, f(x_n)) - (x,y) \| \to 0$

$\Rightarrow \| x_n - x, f(x_n) - y \| \to 0$

$\Rightarrow \| x_n - x \| \to 0 + \| f(x_n) - y \| \to 0$

$\Rightarrow x_n \to x$ & $f(x_n) \to y$

Since f is cont^s. & $x_n \to x \Rightarrow f(x_n) \to f(x)$

$\Rightarrow f(x) = y \Rightarrow (x,y) = (x, f(x)) \in f_G$.

$\Rightarrow f_G$ is closed.

Conversely let f_G be closed. To show that f is cont^s. ?

"Boundednes"

Def. let A be a subset of a topological linear space X over F. We say that A is a bounded if for any neighborhood V of 0 in X, there is a real number $\lambda > 0$ such that $A \subset \lambda V$, and we say that X is locally bounded if there is a bounded neighborhood V of 0 in X.

Note: let A be a subset of a normed space X. A function $f: A \longrightarrow X$ is compact if $f(B)$ is a compact subset of X whenever B is bounded subset of A.

Theorem: let X be a topological linear space X over a field ~~X~~ F and $A, B \subseteq X$. Then

1. If A is finite, then A is bounded.
2. If B is bounded and $A \subseteq B$, then A is bounded.
3. If A and B are bounded sets, then $A \cap B$, $A \cup B$, $A + B$ are bounded sets.
4. If A is bounded, αA is bounded for all $\alpha \in F$.
5. If A is bounded, $\bar{A}$ is bounded.

Proof: (1) Since A is finite set, then $A = \{a_1, \dots, a_n\}$ let V be a neighbor. of 0 in X. Then there exists a balanced neighbor. W of 0 in $X \ni W \subset V$.

Since every neighbor. is absorbing set, then W is absorbing set, so for all $x \in X$ $\exists$ $\lambda > 0 \ni \lambda x \in W$.

since $A \subset X \Rightarrow a_i \in X \ \forall \ i=1,2,\dots,n$

$\Rightarrow \exists \ \lambda_i > 0 \ \ni \ \lambda_i a_i \in W \ \forall \ i=1,2,\dots,n$

Take $\lambda = \max\{\lambda_1, \dots, \lambda_n\}$

Since W balanced set $\Rightarrow \bigcup_{i=1}^{n} \lambda_i W = \lambda W$

$\Rightarrow A \subseteq \bigcup_{i=1}^{n} \lambda_i W \Rightarrow A \subset \lambda W \Rightarrow A \subset \lambda V$

$\Rightarrow A$ is bounded.

2. let V be a neighbor. of 0 in X.

since B is bounded set, then $\exists \ \lambda > 0 \ni B \subset \lambda V$

since $A \subseteq B \Rightarrow A \subset \lambda V \Rightarrow A$ is bounded.

3. (i) sence $A \cap B \subset A$ and A is bounded

$\Rightarrow A \cap B$ is bounded.

(ii) let V be a neighbor. of 0 in X, there is balanced neighbor. W of 0 in X such that $W \subset V$.

since A and B are bounded, then there exists $\lambda_1, \lambda_2 > \ \ni \ A \subset \lambda_1 W$ and $B \subset \lambda_2 W$.

Take $\lambda = \max\{\lambda_1, \lambda_2\}$, since $W \subset V$

$\Rightarrow \lambda W \subset \lambda V \Rightarrow A \cup B \subset \lambda V \Rightarrow A \cup B$ bounded

(iii) Let V be a neighbor. of 0 in X, there is a symmetric neigh. W of 0 in $X \ni W + W \subset V \Rightarrow$ there is a balanced neighbor. U of 0 in X such that $U \subset W$.

Since A, B are bounded, then there exist, $\lambda_1, \lambda_2 > 0 \ \ni$

$A \subset \lambda_1 U$ and $B \subset \lambda_2 U$.

Take $\lambda = \max\{\lambda_1, \lambda_2\}$

since $U \subset W \Rightarrow \lambda(U+U) \subset \lambda(W+W) \subset \lambda V$

$\Rightarrow A + B \subset \lambda V \Rightarrow A + B$ is bounded.

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4. If $\alpha = 0$, then $\alpha A = \{0\} \Rightarrow \alpha A$ is bounded. 58

If $\alpha \neq 0$, let V be a neighbor. of 0 in X, there is a balanced neighbor. W of 0 in $X \ni W \subset V$.

since A is bounded, there is $\lambda > 0 \ni A \subset \lambda W$.

Take $r = \lambda|\alpha| \Rightarrow r > 0$

since W is balanced and $\alpha \leq |\alpha| \Rightarrow \alpha W \subset |\alpha| W$

$\Rightarrow \lambda \alpha W \subset \lambda |\alpha| W$.

since $A \subset \lambda W \Rightarrow \alpha A \subset \lambda \alpha W \subset \lambda|\alpha| W = rW$

since $W \subset V \Rightarrow rW \subset rV \Rightarrow \alpha A \subset rV$

$\Rightarrow \alpha A$ is bounded.

5. Let V be a neighbor. of 0 in X, there is a neighbor. W of 0 in $X \ni \overline{W} \subset V$.

since A is bounded, there is $\lambda > 0$ such that

$A \subset \lambda W \Rightarrow \overline{A} \subset \overline{\lambda W} = \lambda \overline{W}$

we have $\overline{W} \subset V \Rightarrow \lambda \overline{W} \subset \lambda V \Rightarrow \overline{A} \subset \lambda V$.

$\Rightarrow \overline{A}$ is bounded.

Def. let A be a subset of a topological linear space X over F. We say that A is a Totally bounded if for any neighbor. V of 0 in X, there exists a finite subset B of X such that $A \subset B + V$.

Theorem: If A is a totally bounded of a topological linear space over F, then for any neighbor. V of 0 in X, there exists a finite subset A_0 of $A \ni$

$$A \subset V + A_0.$$

Theorem: let X be topological linear space (59)
over a field F and $A, B \subseteq X$. Then
1. If A is finite, then is totally bounded.
2. If A is totally bounded, then A is bounded.
3. If B is a totally bounded and $A \subseteq B$, then A is totally bounded.
4. If A, B are totally bounded sets, then $A \cap B$, $A \cup B$, $A+B$ are totally bounded sets.

Proof: (1) Since $A \subset A + V$, for every neighbor. V of 0 in X. $\Rightarrow$ A is totally bounded.

(2) let V be a neighbor. of 0 in X, there is balanced neighbor. W of 0 in X such that $W \subset V$.
since W is balanced neighbor. of 0 in X and A is totally bounded set, there exists a finite subset B of X such that $A \subset B + W$.

since B is finite $\Rightarrow$ B is bounded, $\Rightarrow$ $\exists\, \alpha > 0 \ni$
$$B \subset \alpha W.$$
Also W balanced $\Rightarrow$ $\alpha W + W \subset (\alpha+1) W$.

Take $\lambda = \alpha + 1 \Rightarrow A \subset \lambda W \subset \lambda V$
$\Rightarrow$ A is bounded set.

3. let V be neighbor. of 0 in X.
since B is a totally bounded, $\exists$ finite subset D of X $\ni B \subset D + V$.
we have $A \subset B \Rightarrow A \subset D + V \Rightarrow A$ is totally bounded.

4. (i) we have $A \cap B \subset A$ & A is totally bounded
$\Rightarrow A \cap B$ is totally bounded by using (3).

(ii) let V be a neighbor. of 0 in X. 60

we have A & B are totally bounded, $\exists$ finite subset, $D_1, D_2 \ni A \subset D_1 + V$ & $B \subset D_2 + V$.

Take $D = D_1 \cup D_2 \Rightarrow D$ is finite subsets and

$A \cup B \subset D \cup V \Rightarrow A \cup B$ is totally bounded.

(iii) let V be a neighbor. of 0 in X. $\exists$ symmetric neighb. W of 0 in $X \ni W + W \subset V$

we have A & B are totally bounded, then there are finite subset D_1, D_2 such that

$A \subset D_1 + W$ and $B \subset D_2 + W$.

Take $D = D_1 \cup D_2 \Rightarrow D$ is finite subset,

$A + B \subset D + W + W \subset D + V \Rightarrow A + B$ is totally bounded.

Def. let X be topological linear space over F.

1. A sequence $\{x_n\}$ in X is said to be Converge to the point $x \in X$ if for every neighbor. V of 0 in X, there exists, $K \in \mathbb{Z}^+ \ni x_n \in x + V \quad \forall n \geq K$ and we write $x_n \to x$ or $\lim_{n \to \infty} x_n = x$

2. A sequence $\{x_n\}$ in X is said to Cauchy sequence if $\forall$ neighbor. V of 0 in X, $\exists K \in \mathbb{Z}^+ \ni x_n - x_m \in V$ for all $n, m \geq K$.